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THE DISCRETE, ORTHOGONAL WAVELET TRANSFORM, A PROJECTIVE APPROACH

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THE DISCRETE, ORTHOGONAL WAVELET TRANSFORM, A PROJECTIVE APPROACH

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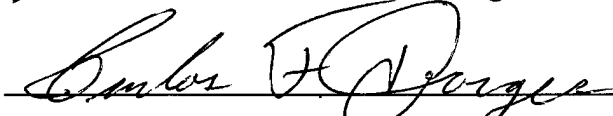
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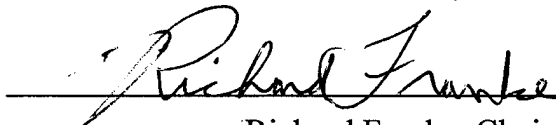
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ABSTRACT

All integral transforms can be viewed as projections onto collections of functions in a Hilbert space. The properties of an integral transform are completely determined by the collection of functions onto which it projects. The wavelet transform projects onto a set of functions which satisfy a simple linear relationship between different levels of dilation. The properties of the wavelet transform are determined by the coefficients of this linear relationship. This thesis examines the connections between the wavelet transform properties and the linear relationship coefficients.

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I.   INTRODUCTION

A great deal has been written about wavelet analysis in the last several years. Much of this work however is presented in the language of signal processing. This is to be expected since wavelet analysis has been primarily developed in the signal processing field. This paper presents the discrete wavelet transform in a projective framework. The discrete wavelet transform, like most integral transforms, can be viewed as projection onto a new basis determined by a given collection of functions. In light of this the properties of the transform are primarily determined by the properties of the basis functions. The development of a transform is then a question of finding a basis which will produce the properties we desire.

What properties do we desire in the wavelet transform? We would like the transform to incorporate the spectral decomposition, or frequency localization, of the Fourier Transform. In addition, we would like to have time-localization of the input function. This allows us to determine when in time a given component of the input function occurs. Frequency localization in the Fourier Transform is accomplished by projecting onto sines and cosines which have non-compact, in fact infinite, support. This precludes the Fourier Transform from having time localization capability. Time localization can be achieved by using the short time Fourier Transform (STFT), which analyzes the input function in small time windows. This time localization is fixed in scale, depending on the time window used. Another desired property of the wavelet transform is arbitrary approximation power. This is useful if we wish to use the transform in a data compression scheme, that is, setting to zero transform coefficients below a certain threshold level and storing the input function in terms of this reduced coefficient set. Finally, and perhaps most importantly, we desire the basis onto which we project to be orthonormal. This will allow easy inversion of the transform. If the basis is orthonormal, inversion is accomplished by weighted superposition of the basis functions using the transform coefficients as weights. This

is the case with the Fourier Transform which uses the orthonormal sine and cosine basis. This is contrasted with the Laplace Transform which uses an exponential basis which is not orthogonal over the real line. This is why the Laplace Transform must be inverted by a contour integral in the complex plane.

The question of developing a transform with these properties is reduced to that of finding a basis for L^2, the space of square integrable functions on the real line, which possesses the desired properties, namely, time and frequency localization, arbitrary approximation power, and orthogonality. If we restrict our attention to compactly supported functions, we can achieve a measure of time localization, however only at a fixed level dependent on the support of the basis function. This severely limits the frequency information available as well, since we can only clearly discern the frequency which corresponds to the basis functions. Compactly supported functions are easier to compute with, hence their use in finite element methods, and we would like to use them here. We need to find a way to use compactly supported basis functions without limiting the time or frequency information available to us. The solution is to use dilations of a compactly supported function. A dilation of $f(x)$ is $f(ax)$ for some a, normally $a > 0$. Notice that as we dilate a function f the support of f changes and also the frequency of f changes. This allows us to cover much wider time and frequency scales than using basis functions of fixed compact support. To give us the maximum amount of flexibility we will use a basis composed of translations and dilations of a fixed function ϕ.

Consider the dilation equation, or two scale recursion relation

$$\phi(x) = \sum_k c_k \phi(2x - k) \qquad \text{for some } \{c_k\}. \qquad (\text{I.1})$$

The solution is completely determined by the dilation equation coefficients $\{c_k\}$. Also notice this dilation equation with coefficients $\{c_0 = 1, c_1 = 1\}$ is satisfied by the Haar function:

$$h(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ 0 & \text{for } x \notin [0,1] \,. \end{cases}$$
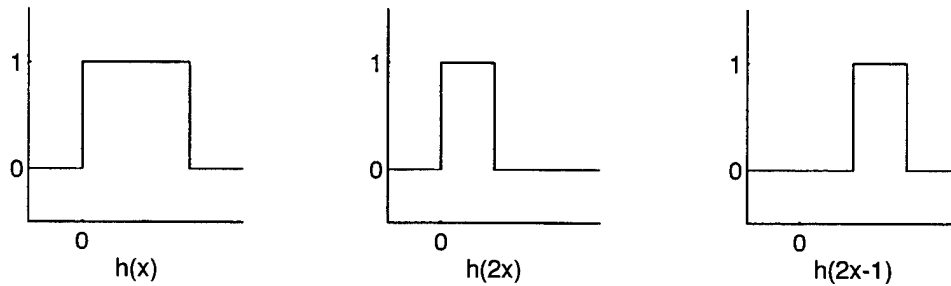
Figure 1. Dilation of Haar Function

The Haar Transform which uses as a basis functions derived from $h(x)$ is in fact an early example of a wavelet transform and predates the current interest in wavelet analysis by several decades.

In this paper we show that solutions to I.1 exist which have arbitrary approximation power and are orthogonal to their integer translates. We further show how these solutions can be used to construct a basis for L^2, and how the discrete transform which represents projection onto this basis can be computed in order N operations.

II.   EXISTENCE, UNIQUENESS, APPROXIMATION, AND ORTHOGONALITY

A.   EXISTENCE AND UNIQUENESS

Before we investigate the relationship between the coefficients of the dilation equation and the properties of the solutions, we need to determine under what conditions a solution in fact exists. To begin our search for solutions to the dilation equation let us consider the iteration

$$\phi_{n+1} = \sum_k c_k \phi_n \left(2x - k\right). \tag{II.1}$$

Solutions to the dilation equation are fixed points of the dynamical system defined by this iteration. The following theorem, which we present without proof, gives conditions under which square summable solutions to the dilation equation exist and is found in [Ref. 1].

THEOREM II.1. *If the $\{c_k\}$ satisfy the following conditions*

1. $\{c_k\}$ is a finite set,

2. $\sum_k c_k = 2$,

3. $\left|m_0\left(z\right)\right|^2 + \left|m_0\left(z + \pi\right)\right|^2 = 1$ for all $z \in \mathbb{R}$, where $m_0\left(z\right) = \sum_k c_k e^{-ikz}$

then the dynamical system in II.1 has a nontrivial square summable fixed point with compact support.

In the same paper, a similar result is presented for distributional solutions to I.1 which does not require the third condition. In practice we are looking for a basis for the function space L^2, so we restrict our attention to square summable solutions.

Uniqueness of this solution, up to normalization, is guaranteed by the same conditions which give us existence. In the square summable case the normalization is $\int \phi dx = 1$. Observe that this normalization requirement is consistent with condition

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2 above since

$$
\begin{aligned}
\int \phi(x)\, dx &= \int \sum_k c_k \phi(2x - k) \\
&= \frac{1}{2} \sum_k c_k \int \phi(y)\, dy \qquad \text{where } y = 2x - k
\end{aligned}
$$

implies $\quad \dfrac{1}{2}\sum_k c_k = 1 \quad$ or $\quad \sum_k c_k = 2.$

Before proceeding with our discussion of approximation let's look at some solutions to I.1 which are familiar to us. In the introduction we saw for coefficients $\{1, 1\}$ the solution to I.1 was the Haar function. Other familiar functions which are solutions to the dilation equation, though we may not recognize them as such, are the Dirac delta function for $\{2\}$ and the cardinal B-splines for $\left\{\binom{n}{k}\right\}$ normalized to sum to two. The Figure 2 shows the dilation of the "Hat" function which is the solution for the dilation equation for $\left\{\frac{1}{2}, 1, \frac{1}{2}\right\}$, the binomial coefficients for $n = 2$, normalized to sum to two.

The Cardinal B-splines illustrate why the third condition in Theorem II.1 is a sufficient but not necessary condition for square summability. The Cardinal B-splines are clearly square summable, but it is easily seen that $\left\{\binom{n}{k}\right\}$ do not satisfy the third condition of Theorem II.1.

B. SOME TOOLS FOR DILATION EQUATIONS

Before proceeding with our investigation of the relationship between the dilation equation coefficients $\{c_k\}$ and the corresponding solution $\phi(x)$, we should introduce a very useful tool in the study of dilation equations. Dilation equations are difficult to study because only in special cases do we actually know what the solution function is. We are usually limited to knowing only the recurrence relation between different scalings of the function. Faced with this lack of information we must turn to the Fourier Transform to glean any information we can about the function. Let us see what the recursive nature of I.1 can tell us about the solution $\phi(x)$.
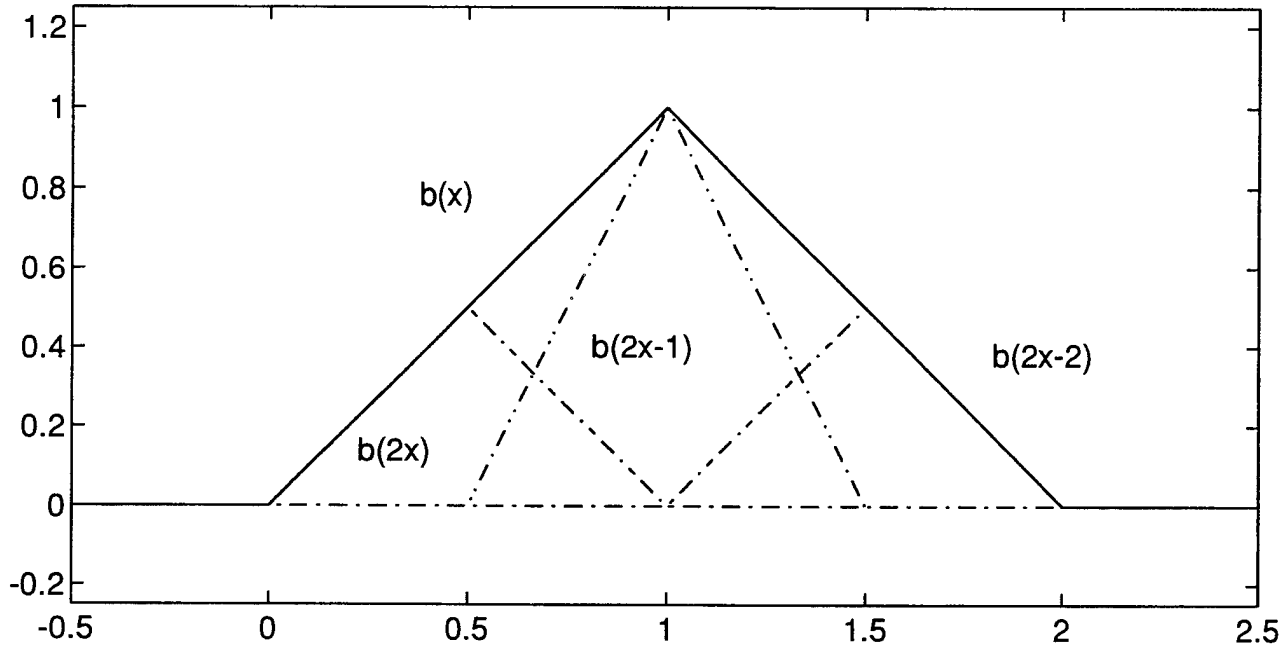
Figure 2. Dilation of "Hat" Function

We are given

$$\phi(x) = \sum_k c_k \phi(2x - k) \qquad \text{for some } \{c_k\}. \tag{II.2}$$

Now, consider the Fourier Transform $\Phi(\xi)$ of $\phi(x)$

$$\Phi(\xi) = \int_{-\infty}^{\infty} \phi(x) e^{-i\xi x} dx \tag{II.3}$$

substituting II.2 into II.3 we get

$$\Phi(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} \sum_k c_k \phi(2x - k) dx$$

or

$$\Phi(\xi) = \sum_k c_k \int_{-\infty}^{\infty} \phi(2x - k) e^{-i\xi x} dx$$

in each integral of this sum make the change of variables $y = 2x - k$ which yields

$$\Phi(\xi) = \frac{1}{2} \sum_k c_k e^{\frac{-i\xi k}{2}} \int_{-\infty}^{\infty} \phi(y) e^{\frac{-i\xi y}{2}} dy. \tag{II.4}$$

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Now, notice the integral in II.4 is $\Phi(\frac{\xi}{2})$,so we have

$$\Phi\left(\xi\right) = \frac{1}{2}\sum_k c_k e^{\frac{-i\xi k}{2}} \Phi\left(\frac{\xi}{2}\right).$$

Applying this process repeatedly to $\Phi(\xi/2), \Phi(\xi/4),...\Phi(\xi/2^n)$ we get

$$\Phi(\xi) = \prod_{j=1}^{n} \frac{1}{2}\sum_k c_k e^{\frac{-i\xi k}{2^j}} \Phi(\frac{\xi}{2^n}).$$

Let us now take the limit as $n \to \infty$,

$$\Phi(\xi) = \prod_{j=1}^{\infty} \frac{1}{2}\sum_k c_k e^{\frac{-i\xi k}{2^j}} \Phi(0). \tag{II.5}$$

If we require by way of normalization that $\int \phi dx = 1$ then we have $\Phi(0) = \int \phi dx = 1$ and II.5 becomes

$$\Phi\left(\xi\right) = \prod_{j=1}^{\infty} \frac{1}{2}\sum_k c_k e^{\frac{-i\xi k}{2^j}}.$$

Notice we could write this infinite product as

$$\Phi\left(\xi\right) = \prod_{j=1}^{\infty} \frac{1}{2}\sum_k m_0\left(\frac{\xi}{2^j}\right).$$

Now we see the importance of the $m_0(z)$ expression in the third condition of Theorem II.1, the behavior of this trigonometric polynomial determines the convergence of the infinite product representation of the Fourier Transform, and therefore the existence of a square summable solution to I.1. A fairly simple, yet powerful result of this infinite product representation of the Fourier Transform is the Convolution Theorem for Dilation Equations.

THEOREM II.2. *Let ϕ and φ be solutions to the dilation equation for coefficients $\{c_k\}$ and $\{d_l\}$ respectively. The function $\phi * \varphi$ given by the continuous convolution of ϕ with φ is the solution to the dilation equation with coefficients $\{b_j\}$, where $\{b_j\}$ is one half the discrete convolution of $\{c_k\}$ with $\{d_l\}$.*

Proof. As we have seen the Fourier Transforms of ϕ and φ are given by

$$\Phi\left(\xi\right) = \prod_{n=0}^{\infty} \frac{1}{2}\sum_k c_k e^{\frac{-i\xi k}{2^n}} \qquad \text{and} \qquad \Psi\left(\xi\right) = \prod_{n=0}^{\infty} \frac{1}{2}\sum_l d_l e^{\frac{-i\xi l}{2^n}}$$

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respectively. The product of these transforms $\Phi\Psi$ is given by

$$\Phi\Psi = \prod_{n=0}^{\infty} \frac{1}{2} \sum_{k} c_k e^{\frac{-i\xi k}{2^n}} \prod_{m=0}^{\infty} \frac{1}{2} \sum_{l} d_l e^{\frac{-i\xi l}{2^m}}.$$

We can combine the two product operators by associating the factors where $n = m$ to get

$$\Phi\Psi = \prod_{n=0}^{\infty} \frac{1}{2} \sum_{k} c_k e^{\frac{-i\xi k}{2^n}} \frac{1}{2} \sum_{l} d_l e^{\frac{-i\xi l}{2^n}}.$$

Observe each factor in this infinite product is a product of two polynomials in $e^{\frac{-i\xi}{2^n}}$, namely, $\frac{1}{2} \sum_{k} c_k e^{\frac{-i\xi k}{2^n}}$ and $\frac{1}{2} \sum_{l} d_l e^{\frac{-i\xi l}{2^n}}$. Let $p\left(\xi\right)$ be the product of these polynomials. From polynomial multiplication, we see the coefficients of $p\left(\xi\right)$ are given by

$$b_\lambda = \frac{1}{2} \sum_{j=0}^{\lambda} c_j d_{\lambda-j},$$

that is, the set $\{b_\lambda\}$ is one half the discrete convolution of $\{c_k\}$ with $\{d_l\}$, and the product polynomial p is

$$p(\xi) = \frac{1}{2} \sum_{\lambda} b_\lambda e^{\frac{-\xi\lambda}{2^n}}.$$

Substituting this into the expression for $\Phi\Psi$ gives us

$$\Phi\Psi = \prod_{n=0}^{\infty} \frac{1}{2} \sum_{\lambda} b_\lambda e^{\frac{-i\xi\lambda}{2^n}}.$$

Observe the right hand side of this equation is the infinite product representation of the Fourier Transform of the solution to the dilation equation with coefficients $\{b_\lambda\}$. Thus, $\Phi\Psi$ is the Fourier Transform of the solution to the dilation equation for coefficients $\{b_j\}$. Now, from the convolution theorem for Fourier Transforms we know $\Phi\Psi$ is the transform of $\phi * \varphi$, the continuous convolution of ϕ with φ. Since Fourier Transforms are unique this completes the proof. $\square$

Armed with this result we see why the Cardinal B-splines are solutions to the dilation equation for the binomial coefficients normalized to sum to two. If we develop B-splines from a repeated convolution standpoint as is done in [Ref. 4], the

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relationship between B-splines and binomial coefficients becomes transparent in light of Theorem II.2. To generate the B-splines we start with the Haar function which we can consider the "zeroth" order B-spline (piecewise constant approximation), we get the next higher order B-spline (piecewise linear approximation) by convolution with the Haar function. By our convolution theorem this says we should take one half the discrete convolution of $\{1, 1\}$ with itself. Convolving $\{1, 1\}$ with itself we get $\{1, 2, 1\}$ the binomial coefficients for $n = 2$, which we must normalize to sum to two. Similarly, to generate the $(n + 1)^{st}$ degree B-spline we convolve the nth degree B-spline with the Haar function, so we convolve the coefficients of the nth degree B-spline with $\{1, 1\}$. This gives us the binomial coefficients for $n+1$, which again we must normalize to sum to two. The appearance of the B-splines is of more than passing interest. B-splines and their translates are often used as a basis for approximation. This is very similar to the scheme we are investigating. We are pursuing the goal of representing L^2 in its entirety rather than approximating functions in a subspace of L^2. Another aspect of spline approximation which will be of great importance in the next section is the use of sliding window filters as a tool to smooth data. Sliding window filters can be viewed as convolution with the Haar function. Thus repeated application of sliding window filters is equivalent to repeated convolution with the Haar function, both of which lead to smoother data, or smoother approximation if one is constructing a basis function, as we shall see in the next section.

C. APPROXIMATION

Since we will be using solutions to I.1 as a basis for approximation we would like to know how well an arbitrary function can be approximated by ϕ and its translates. Our measure of approximation power will be polynomial precision, or what degree arbitrary polynomial can be exactly represented by ϕ and its translates. A key step in finite element methods is to approximate a function by the translates of a compactly supported basis function. To determine the approximation power of solutions to the

dilation equation we consider the work done on the finite element method by Strang and Fix. In [Ref. 3] they show that the following statements are equivalent:

1. Any polynomial of degree less than or equal to $p-1$ can be exactly represented by a linear combination of ϕ and its translates.

2. $\Phi(0) \neq 0$ and $\frac{d^\alpha}{d\xi^\alpha}\Phi(\xi)\mid_{\xi=2\pi n}= 0$ for $n \in Z, n \neq 0$ and $\alpha = 0, 1, ..., p$.

The first of these statements is the definition of approximation power that we are interested in, the second is related to the Fourier Transform of the function ϕ. We will use the second statement to construct a condition on $\{c_k\}$ that will give the desired approximation power. First notice that $\Phi(0) \neq 0$ is taken care of by our normalization requirement since

$$\Phi(0) = \int \phi e^{-i0x} dx = \int \phi dx = 1.$$

To satisfy the rest of this statement we must look at the infinite product representation of $\Phi(\xi)$

$$\Phi(\xi) = \prod_{n=1}^{\infty} \frac{1}{2} \sum_k c_k e^{\frac{-i\xi k}{2^n}}.$$

Since this product converges and $\Phi(0) \neq 0$ we see that the product vanishes for some value of ξ if and only if at least one of its factors vanish for that value of ξ. Observe that the following condition is sufficient to ensure the desired behavior of $\Phi(\xi)$:

- $z = \pi$ is a root of order p of $m_0(z)$.

In the previous section we saw

$$\Phi(\xi) = \prod_{n=1}^{\infty} m_0\left(\frac{\xi}{2^n}\right). \tag{II.6}$$

Now, for all $l \in Z, l \neq 0$, there exists n such that $\frac{2\pi l}{2^n}$ mod $2\pi = \pi$. Therefore, one of the factors in II.6 is zero, and $\Phi(2\pi l) = 0$. Now consider $\frac{d}{d\xi}[\Phi(\xi)]$,

$$\frac{d}{d\xi}[\Phi(\xi)] = \frac{d}{d\xi}\left[m_0\left(\frac{\xi}{2^n}\right)\right]\prod_{k \neq n} m_0\left(\frac{\xi}{2^k}\right) + m_0\left(\frac{\xi}{2^n}\right)\frac{d}{d\xi}\left[\prod_{k \neq n} m_0\left(\frac{\xi}{2^k}\right)\right]. \tag{II.7}$$

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Since $m_0(z)$ has a root of order p at π, the right hand side of II.7 is zero. Similarly we can see $\frac{d^{(j)}}{d\xi^j}[\Phi(\xi)] = 0$ for $j = 0, 1, ..p$ provided $m_0(z)$ has a zero of order p at π.

Unfortunately, this elegant sufficient condition on $m_0(z)$ is **not** necessary. Observe, if $\frac{d}{d\xi}[\Phi(\xi)] = 0$, $\frac{d}{d\xi}[m_0(\pi)]$ need not be zero, since $m_0\left(\frac{\pi}{2}\right)$, which is a factor of $\prod_{k \neq n} m_0\left(\frac{\xi}{2^k}\right)$ could be zero, in which case, II.7 will be satisfied regardless of the value of $\frac{d}{d\xi}[m_0(\pi)]$.

The sufficient condition for the approximation power of ϕ can be arrived at in several ways. In [Ref. 2] it is presented as a sum condition, that is:

$$\sum_{l=0}^{N} (-1)^l l^k c_l = 0 \qquad for \ k = 0, 1, ..., p - 1. \qquad (\text{II.8})$$

We can see this is equivalent to our condition that π be a root, of order p, of $m_0(z)$, by first expressing the sum condition as a matrix equation,

$$\mathbf{Kc = 0}$$

where

$$K_{i,j} = (-1)^j j^i.$$

$\mathbf{K}$ is a $p \times N + 1$ matrix with the following structure:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \cdots \\ 0 & -1 & 2 & -3 & 4 & \cdots \\ 0 & -1 & 4 & -9 & 16 & \cdots \\ 0 & -1 & 8 & -27 & 64 & \cdots \\ 0 & -1 & 16 & -81 & 256 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Calculating the null space of this matrix yields a basis for the null space which is given by $\left\{ \left[v_p^k \right] \right\}$ where $\left[v_p^k \right]$ is the $N + 1$ long vector whose non-zero entries begin at the kth position and are the binomial coefficients $\binom{p}{j}$ for $k = 1, 2, ..., N - p$. For

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example, a basis for the null space of

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & -1 & 2 & -3 & 4 \\ 0 & -1 & 4 & -9 & 16 \end{bmatrix}$$

is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Since this is a basis for the null space of $\mathbf{K}$, any $\{c_k\}$ which satisfies the sum condition II.8 must be a linear combination of the basis vectors. Since the basis vectors are shifted copies of one vector, the matrix whose columns are the basis vectors is Toeplitz. So, any linear combination of the basis vectors can be represented as the product of this Toeplitz matrix $\mathbf{T}$ with a vector whose elements are the weights in the linear combination. Alternatively, we can consider this multiplication by a Toeplitz matrix to be discrete convolution of the vector of weights with the vector of binomial coefficients, i.e. polynomial multiplication if we consider the elements of these vectors to be coefficients of polynomials.

So, what does all this do for us? Any $\{c_k\}$ which satisfies II.8 can be considered to be the coefficients of a polynomial $f(x)$ which in turn can be expressed as the product of two polynomials $g(x)$ and $h(x)$. The coefficients of $g(x)$ are given by the binomial coefficients $\binom{p}{j}$ where p is the required approximation power. Recall that the polynomial with coefficients $\binom{p}{j}$ is $g(x) = (x+1)^p$. That is, the polynomial with coefficients $\{c_k\}$ has a root of order p at $x = -1$. Now, if we consider the relationship between the trigonometric polynomial $m_0(z) = \sum_k c_k e^{ikz}$ and the polynomial $f(x) = \sum_k c_k x^k$, we see $f(e^{iz})$ is equivalent to $m_0(z)$. Further, if $f(x)$ has a root at $x = -1$, then $m_0(z)$ has a root at π.

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Yet another way to see the sufficiency of our condition for approximation power is to observe that each higher order of the root of $m_0(z)$ at π is equivalent to an additional factor of $(x+1)$ in the polynomial whose coefficients are $\{c_k\}$. In turn, each of these factors is equivalent to the discrete convolution of a smaller set of coefficients with $\{1,1\}$. Now, by our convolution theorem for dilation equations each of these discrete convolutions is equivalent to the continuous convolution of a function which precedes ϕ with the function whose dilation equation coefficients are $\{1,1\}$. Of course the function with dilation equation coefficients $\{1,1\}$ is the Haar function. Recall that the Fourier Transform of the Haar function is $\frac{e^{\frac{-i\xi}{2}} \sin(\xi)}{\xi}$, which decays like $\frac{1}{\xi}$. Each factor of $(x+1)$ in the polynomial whose coefficients are $\{c_k\}$ introduces a factor which decays like $\frac{1}{\xi}$ in the Fourier Transform of ϕ. Each factor of $\frac{1}{\xi}$ in the Fourier Transform implies one degree of smoothness, or approximation power in the function ϕ.

D. ORTHOGONALITY

As stated in the introduction, an orthogonal basis of functions is advantageous. Orthogonality allows us to reconstruct the input function from its transform coefficients by summing the basis functions weighted by the transform coefficients. This is the case in the Fourier Transform. By contrast the Laplace Transform uses the exponential functions $\{e^{px}\}$ as a basis. These functions are not orthogonal over the real line and as a result the inverse Laplace Transform involves a contour integral in the complex plane. We wish to take advantage of this ease of inversion, so we will restrict our attention to orthogonal basis functions. Again, we are limited by not having an explicit representation of our basis functions. We must show that there exist solutions to I.1 which are orthogonal to their integer translates. Additionally, we still require these functions to have the desired approximation power.

Fortunately, such solutions do exist. In fact, we have already seen one of them. The Haar function is the solution to the dilation equation with coefficients

$\{1, 1\}$. Since the Haar function has support width of one unit, any integer translate of the Haar function has support which does not intersect the support of the original Haar function therefore the Haar function must be orthogonal to any of its integer translates. However, the Haar function has poor approximation power, and is not an ideal candidate for use as a basis for an integral transform. The other members of the Cardinal B-spline family are used for piecewise approximation by higher order polynomials and therefore have increasing approximation power and might be good candidates for transform basis functions. Unfortunately, the higher order Cardinal B-splines are not orthogonal. This is easily seen since each of the B-splines is positive valued over their support and the product of two B-splines with intersecting support would be positive, thus the inner product of these B-splines would be non-zero. In this section we develop both necessary and sufficient conditions for orthogonal solutions to the dilation equation.

Let us first look at the consequences of this orthogonality, and develop a necessary condition. Suppose $\phi(x)$ is orthogonal to its integer translates $\phi(x - k)$ for all $k \in \mathbb{Z}$, $k \neq 0$. Orthogonality here is with respect to the inner product on $\mathbf{L}^2$, that is

$$\langle f(x), g(x) \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx.$$

So far we have only considered real dilation equation coefficients. We will continue to restrict our attention to real coefficients. However, the following development is presented allowing the possibility of complex coefficients to be consistent with the above inner product.

Suppose $\phi(x)$ is orthogonal to $\phi(x - k)$ for $k \in \mathbb{Z}$, $k \neq 0$; additionally suppose $\phi(2x)$ is orthogonal to $\phi(2x - j)$ for $j \in \mathbb{Z}$, $j \neq 0$, then

$$0 = \langle \phi(x), \phi(x - k) \rangle = \int_{-\infty}^{\infty} \phi(x) \overline{\phi(x - k)} dx.$$

Substituting in I.1 for $\phi(x)$ and $\phi(x - k)$ we get

$$\int_{-\infty}^{\infty} \phi(x) \overline{\phi(x - k)} dx = \int_{-\infty}^{\infty} \sum_{l} c_l \phi(2x - l) \sum_{m} \overline{c_m} \overline{\phi(2(x - k) - m)} dx.$$

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Interchanging the order of integration and finite summation

$$\int_{-\infty}^{\infty} \phi(x) \overline{\phi(x-k)} dx = \sum_l \sum_m c_l \overline{c_m} \int_{-\infty}^{\infty} \phi(2x-l) \overline{\phi(2x-2k-m)} dx. \qquad (II.9)$$

Since $\phi(2x)$ is orthogonal to $\phi(2x-j)$ for $j \in \mathbb{Z}$, $j \neq 0$ the integral on the right hand side of II.9 must be zero unless $l = 2k + m$. This simplifies II.9 to

$$\int_{-\infty}^{\infty} \phi(x) \overline{\phi(x-k)} dx = \sum_l c_l \overline{c_{l-2k}} \int_{-\infty}^{\infty} \phi(2x-l) \overline{\phi(2x-l)} dx.$$

So, if $\phi(x)$ is to be orthogonal to $\phi(x-k)$ we require

$$\sum_l c_l \overline{c_{l-2k}} \int_{-\infty}^{\infty} \phi(2x-l) \overline{\phi(2x-l)} dx = 0.$$

And since we assume ϕ is not identically zero, the above integral cannot be zero, thus

$$\sum_l c_l \overline{c_{l-2k}} = 0 \qquad \text{for } k \in \mathbb{Z}, k \neq 0. \qquad (II.10)$$

Additionally, when $k = 0$

$$\int_{-\infty}^{\infty} \phi(x) \overline{\phi(x)} dx = \sum_l c_l \overline{c_l} \int_{-\infty}^{\infty} \phi(2x-l) \overline{\phi(2x-l)} dx.$$

Let $y = 2x - l$ in the right hand integral and we get

$$\int_{-\infty}^{\infty} \phi(x) \overline{\phi(x)} dx = \frac{1}{2} \sum_l c_l \overline{c_l} \int_{-\infty}^{\infty} \phi(y) \overline{\phi(y)} dy.$$

For this equation to be true we must have

$$\sum_l c_l \overline{c_l} = 2.$$

We can combine these to get a necessary condition for orthogonality

$$\sum_l c_l \overline{c_{l-2k}} = 2\delta_k \qquad \text{for all } k \in \mathbb{Z} \qquad (II.11)$$

where δ_k is Kronecker's delta function: $\delta_k = 1$ for $k = 0$, $\delta_k = 0$ for $k \neq 0$.

It appears that II.10 is yet another condition we are imposing on the dilation equation coefficients. However, with a little work we will see that this necessary

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condition for orthogonality in II.10 is equivalent to the third sufficient condition for square summability in Theorem II.1, expressed in terms of the coefficients rather than in terms of the polynomial $m_0(z)$. Let us start with the third condition from Theorem II.1,

$$|m_0(z)|^2 + |m_0(z+\pi)|^2 = 1 \qquad \text{for all } z \in \mathbb{R} \qquad \text{(II.12)}$$

$$\text{where} \qquad m_0(z) = \frac{1}{2}\sum_k c_k e^{ikz}. \qquad \text{(II.13)}$$

First, observe

$$m_0(z+\pi) = \frac{1}{2}\sum_k c_k e^{ik(z+\pi)} = \frac{1}{2}\sum_k c_k e^{ikz} e^{ik\pi}. \qquad \text{(II.14)}$$

Now, if k is even, $e^{ik\pi} = 1$, and if k is odd, $e^{ik\pi} = -1$ so

$$m_0(z+\pi) = \frac{1}{2}\sum_k (-1)^k c_k e^{ikz}.$$

Recall $|\zeta|^2 = \zeta\overline{\zeta}$ which leads us to

$$|m_0(z)|^2 + |m_0(z+\pi)|^2 = m_0(z)\overline{m_0(z)} + m_0(z+\pi)\overline{m_0(z+\pi)}.$$

Substituting in the expression for $m_0(z)$ from II.13 and the expression for $m_0(z+\pi)$ from II.14 we now get

$$|m_0(z)|^2 + |m_0(z+\pi)|^2 =$$
$$\frac{1}{2}\sum_k c_k e^{ikz} \overline{\frac{1}{2}\sum_l c_l e^{ilz}} + \frac{1}{2}\sum_k (-1)^k c_k e^{ikz} \overline{\frac{1}{2}\sum_l (-1)^l c_l e^{ilz}}. \qquad \text{(II.15)}$$

Carrying the conjugation down into the trigonometric polynomials and simplifying we get

$$|m_0(z)|^2 + |m_0(z+\pi)|^2 =$$
$$\frac{1}{4}\sum_k c_k e^{ikz} \sum_l \overline{c_l} e^{-ilz} + \frac{1}{4}\sum_k (-1)^k c_k e^{ikz} \sum_l (-1)^l \overline{c_l} e^{-ilz}. \qquad \text{(II.16)}$$

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This sum is a Laurent polynomial in e^{iz}. Keeping in mind that this sum must equal one for all $z \in \mathbb{R}$, we see that the coefficients of $(e^{iz})^n$ for $n \neq 0$ must be zero since the sum is real-valued. We need an expression for the coefficient of $(e^{iz})^n$; with a little work we can see this expression is

$$b_n = \frac{1}{4} \sum_j c_j \overline{c_{j-n}} + \frac{1}{4} (-1)^n \sum_j c_j \overline{c_{j-n}}.$$

Now notice, if n is odd then b_n is zero automatically. If n is even then we can say $n = 2m$ for some $m \in \mathbb{Z}$, and

$$b_n = \frac{1}{2} \sum_j c_j \overline{c_{j-2m}} = 0$$

thus

$$\sum_j c_j \overline{c_{j-2m}} = 0 \qquad \text{for } m \in \mathbb{Z} , \ m \neq 0$$

when $m = 0$, $b_n = 1$ and

$$\sum_j c_j \overline{c_j} = 2.$$

Combining these conditions we get

$$\sum_j c_j \overline{c_{j-2m}} = 2\delta_m \qquad \text{for all } \quad m \in \mathbb{Z}$$

as in condition II.10. This chain of arguments can be reversed to derive II.12 from II.10. So, we have shown the equivalence of the two conditions. If we look for orthogonal solutions to the dilation equation, the coefficients necessarily satisfy the sufficient condition for square summability.

This is all well and good, but we still need to find a sufficient condition on the dilation equation coefficients to ensure orthogonality. There is a necessary and sufficient condition for orthogonality of the solutions of the dilation equation. This condition can be found in [Ref. 5], where it is expressed in terms of the location of the roots of the trigonometric polynomial $m_0(z)$ as defined in Theorem II.1. This

condition, while it has the advantage of being both necessary and sufficient, is a little cumbersome. We will instead present a slightly stronger condition on the placement of roots of $m_0(z)$ which is also sufficient for orthogonality. This condition is found in [Ref. 5] as well and relies on the necessary and sufficient condition mentioned above. We present it here and refer the reader to [Ref. 5] for proof.

THEOREM II.3. *Let ϕ be the solution to the dilation equation for coefficients $\{c_k\}$. Suppose $m_0(z) = \frac{1}{2}\sum_k c_k e^{ikz}$ satisfies $|m_0(z)|^2 + |m_0(z+\pi)|^2 = 1$ for all $z \in \mathbb{R}$, and $m_0(0) = 1$. If $m_0(z)$ has no zeros in $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, then $\phi(x)$ is orthogonal to $\phi(x-k)$ for all $k \in \mathbb{Z}$, $k \neq 0$.*

Notice since we required $\sum_k c_k = 2$, we automatically satisfy $m_0(0) = 1$. Also, the sufficient condition for approximation power was that $m_0(z)$ have roots of a given multiplicity at π, it is encouraging that this does not conflict with the sufficient condition for orthogonality.

Well known examples of compactly supported, orthogonal solutions to the dilation equation with higher approximation power than the Haar function are the scaling functions associated with the Daubechies' Wavelets [Ref. 2]. We will describe what a wavelet is in the next chapter.

III.   THE DISCRETE ORTHOGONAL WAVELET TRANSFORM

A.   DECOMPOSITION OF THE FUNCTION SPACE L^2

In the previous chapter we saw that we could find solutions to the dilation equation. We also saw that these functions could be made to have a specified approximation power and be orthogonal to their integer translates. In order to develop a transform based on these functions we must show that a basis for L^2, the space of square integrable functions defined on the real numbers, can be formed from solutions to the dilation equation. As we have previously seen, the solutions to the dilation equation are square integrable provided the dilation equation coefficients meet some basic conditions. In this section we will show that we can derive from the solution to the dilation equation a basis for L^2, and that we need not impose any additional conditions on the dilation equation coefficients to accomplish this.

Let us consider the space, call it V_0, of functions which can be expressed as linear combinations of the solution ϕ to the dilation equation and its integer translates. Since ϕ is square integrable, V_0 is a subspace of L^2. Let us also consider the space, call it V_{-1}, spanned by $\phi(2x - k)$ where $k \in \mathbb{Z}$, this space is composed of functions which are linear combinations of $\sqrt{2}\phi(2x)$ and its half integer translates. We have scaled the function by $\sqrt{2}$ so that the norm, in L^2, of the prospective basis function will be one, this has no impact on the space spanned by the functions. Again, since ϕ is square integrable $\phi(2x)$ is also square integrable and V_{-1} is a subspace of L^2. If we continue in this manner, taking spaces spanned by $\phi_{jk} = 2^{-j/2}\phi(2^{-j}x - k)$, we get an infinite sequence of subspaces of L^2, namely $V_0, V_{-1}, V_{-2}, \ldots$. If we also allow positive indices for these spaces, which implies negative powers of 2 in $2^{-j/2}\phi(2^{-j}x - k)$ we get a bi-infinite sequence of subspaces of L^2

$$V_j = \text{linear span } \left\{ \phi\left(2^{-j}x - k\right) \mid \quad k \in \mathbb{Z} \right\} \qquad \text{for } j \in \mathbb{Z}.$$

This sequence of subspaces has the following properties.

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1. The V_j form a nested sequence, that is $V_j \subset V_{j-1}$.

Suppose $f \in V_j$ then f is a linear combination of $\phi(2^{-j}x - k)$. Since ϕ is a solution to the dilation equation, $\phi(2^{-j}x - k)$ is a linear combination of $\phi\left(2^{-(j-1)}x - k\right)$, thus $V_j \subset V_{j-1}$.

2. $f(x) \in V_j$ if and only if $f(2x) \in V_{j-1}$.

Suppose $f(x) = \sum \alpha_k \phi(2^{-j}x - k)$, then $f(2x) = \sum \alpha_k \phi(2^{-j}(2x) - k) = \sum \alpha_k \phi\left(2^{-(j-1)}x - k\right)$ thus $f(2x) \in V_{j-1}$. Conversely, suppose $f(2y) \in V_{j-1}$ then, $f(2y) = \sum \alpha_k \phi\left(2^{-(j-1)}y - k\right) = \sum \alpha_k \phi(2^{-j}(2y) - k)$. Now let $x = 2y$, and we get $f(x) = \sum \alpha_k \phi(2^{-j}x - k)$, thus $f(x) \in V_j$.

3. $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.

It is sufficient to show that for all $f \in C_0^\infty$, the space of infinitely differentiable, compactly supported functions, $\langle P_j f, f \rangle \to 0$ as $j \to \infty$. Here $P_j f$ is the orthogonal projection of f onto V_j. As a consequence of II.11 the ϕ_{jk} form an orthogonal set. Thus, $\{\phi_{jk}\}$ constitutes an orthonormal basis for V_j, and the orthogonal projector P_{-j} can be expressed as

$$P_j f = \sum_k \langle \phi_{jk}, f \rangle \phi_{jk}.$$

So

$$\langle P_j f, f \rangle = \left\langle \sum_k \langle \phi_{jk}, f \rangle \phi_{jk}, f \right\rangle = \sum_k |\langle \phi_{jk}, f \rangle|^2$$

and if we assume f has support $[-a, a]$

$$|\langle \phi_{jk}, f \rangle|^2 = \left| \int_{-a}^a f(x) 2^{-j/2} \overline{\phi(2^{-j}x - k)} dx \right|^2$$

$$= 2^{-j} \left| \int_{-a}^a f(x) \overline{\phi(2^{-j}x - k)} dx \right|^2.$$

As a consequence of Hölder's inequality

$$\int_{-a}^a f(x) \overline{\phi(2^{-j}x - k)} dx \le \|f\| \left(\int_{-a}^a \phi(2^{-j}x - k) \overline{\phi(2^{-j}x - k)} dx \right)^{1/2},$$

so

$$2^{-j} \left| \int_{-a}^a f(x) \overline{\phi(2^{-j}x - k)} dx \right|^2 \le 2^{-j} \|f\|^2 \int_{-a}^a \left| \phi(2^{-j}x - k) \right|^2 dx$$

$$\le \|f\|^2 \int_{|y+k| \le 2^{-j}a} |\phi(y)|^2 dy.$$

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Observe that as $j \to \infty$ the integral on the right hand side goes to zero independent of k. Also, each $|\langle \phi_{jk}, f \rangle|^2$ is non-negative. Combining these two facts we see

$$0 \le \langle P_j f, f \rangle \le \|f\|^2 \sum_k \int_{|y+k| \le 2^{-j}a} |\phi(y)|^2 \, dy \to 0 \text{ as } j \to \infty$$

thus, $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.

These properties, while very helpful, leave us somewhat short of our goal of producing a basis for L^2. The most obvious shortcoming is that we have not shown that any collection of these functions span L^2. Before we address the issue of spanning L^2, let us consider a property of Φ, the Fourier Transform of ϕ. As we shall see, the following limit is of interest to us

$$\lim_{j \to \infty} \int_{-\infty}^{\infty} \left| \Phi\left(2^{-j}\omega\right) \right|^2 |g(\omega)|^2 \, d\omega$$

for arbitrary $g \in C_0^\infty$. In [Ref. 1] we see that subject to the conditions already imposed on $\{c_k\}$, Φ is an entire function. Additionally, we have seen $\Phi(0) = 1$, and $|\Phi(z)| \le 1$ for $z \in \mathbb{R}$. We can now use Lebesgue's bounded convergence theorem [Ref. 6] to show

$$\lim_{j \to \infty} \int_{-\infty}^{\infty} \left| \Phi\left(2^{-j}\omega\right) \right|^2 |g(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |g(\omega)|^2 \, d\omega. \tag{III.1}$$

Now, let us get back to trying to span L^2 with some collection of ϕ_{jk}. What we wish to show is $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2$. Since the V_j are nested, it will be sufficient to show that $\langle P_{-j} f, f \rangle \to \langle f, f \rangle$ as $j \to \infty$ for all $f \in L^2$ such that $\widehat{f}$, the Fourier Transform of f, is in C_0^∞. Let $f \in L^2$ such that $\widehat{f} = g \in C_0^\infty$. As we have seen before the orthogonal projector P_{-j} can be expressed as

$$P_{-j} f = \sum_{k \in \mathbb{Z}} \langle f, \phi_{-jk} \rangle \phi_{-jk}.$$

Again we see

$$\langle P_{-j} f, f \rangle = \sum_{k \in \mathbb{Z}} |\langle f, \phi_{-jk} \rangle|^2.$$

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Recalling that the Fourier Transform we are using has the normalization constant in the inverse transform, inner products in the function space are related to inner products in the transform space by $\langle \phi, \psi \rangle = 2\pi \langle \Phi, \Psi \rangle$. By standard operational rules for Fourier Transforms we can see Φ_{-jk}, the Fourier Transform of ϕ_{-jk} as defined above is given by

$$\Phi_{-jk} = e^{-2^j ki\omega} 2^{-j/2} \Phi\left(2^{-j}\omega\right).$$

All of this now leads us to

$$\langle f, \phi_{-jk} \rangle = 2\pi \langle g, \Phi_{-jk} \rangle = 2\pi \int_{-\infty}^{\infty} g(\omega) \overline{2^{-j/2} e^{2^{-j} ki\omega} \Phi\left(2^{-j}\omega\right)} d\omega.$$

Simplifying where possible and making the change of variable $z = 2^{-j}\omega$ we get

$$\langle f, \phi_{-jk} \rangle = 2\pi 2^{-j/2} \int_{-\infty}^{\infty} g\left(2^j z\right) \Phi(z) e^{-kiz} dz.$$

Recall that g has compact support, so for sufficiently large j, $g\left(2^j z\right)$ is nonzero only in $[-\pi, \pi]$ and we can simplify the integral to

$$\langle f, \phi_{-jk} \rangle = 2\pi 2^{-j/2} \int_{-\pi}^{\pi} g\left(2^j z\right) \Phi(z) e^{-kiz} dz. \tag{III.2}$$

Now, the right hand side of III.2 is just the k_{th} Fourier coefficient of the function $2^{-j/2} 2\pi g\left(2^j z\right) \Phi(z)$. The quantity we are interested in, namely $\langle P_{-j} f, f \rangle$, is given by the sum over k of the squares of the magnitudes of the individual inner products given in III.2. Since each of these individual inner products is a Fourier coefficient for a specific function, we can use Parseval's Equality to arrive at

$$\begin{aligned} \langle P_{-j} f, f \rangle &= \sum_k \left| 2\pi 2^{-j/2} \int_{-\pi}^{\pi} g\left(2^j z\right) \Phi(z) e^{-kiz} dz \right|^2 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 2^{-j/2} 2\pi g\left(2^j z\right) \Phi(z) \right|^2 dz. \end{aligned} \tag{III.3}$$

Collecting terms and simplifying where possible we get

$$\langle P_{-j} f, f \rangle = 2\pi 2^{-j} \int_{-\pi}^{\pi} \left| g\left(2^j z\right) \Phi(z) \right|^2 dz.$$

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If we now revert to the original variables, that is, let $2^j z = \omega$ we get

$$\langle P_{-j}f, f \rangle = 2\pi \int_{-2^j \pi}^{2^j \pi} \left| g(\omega) \Phi \left(2^{-j} \omega \right) \right|^2 d\omega = 2\pi \int_{-\infty}^{\infty} \left| g(\omega) \right|^2 \left| \Phi \left(2^{-j} \omega \right) \right|^2 d\omega.$$

We have seen in III.1 that the right hand integral above converges to

$$2\pi \int_{-\infty}^{\infty} \left| g(\omega) \right|^2 d\omega.$$

Combining all we have done we get

$$\lim_{j \to \infty} \langle P_{-j}f, f \rangle = 2\pi \int_{-\infty}^{\infty} \left| g(\omega) \right|^2 d\omega = 2\pi \langle g, g \rangle = \langle f, f \rangle.$$

Which shows $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2$.

So far, we have shown the existence of a sequence of orthonormal sets of functions the limit of which is dense in L^2. It might appear that we are nearing the end of our search for basis functions on which to base our wavelet transform. What we have is a good foundation for an approximation scheme, however this scheme falls short of satisfying all the conditions we need for our basis functions. Namely, projecting onto V_j for fixed j doesn't provide any frequency information. If we attempt to get frequency information by projecting onto other subspaces V_n for various $n \neq j$, we no longer have a orthonormal system, since ϕ_{jk} is not orthogonal to ϕ_{nm} for $j \neq n$ as either $V_j \subset V_n$ or $V_n \subset V_j$. This leads us to introduce the detail space W_j associated with V_j, which we will call the scaling space. Formally, W_j is the orthogonal complement of V_j in V_{j-1}. This allows us to express V_{j-1} as the direct sum of V_j and W_j. We can express this graphically as

$$V_j \;\; \rightarrow \;\; V_{j+1} \rightarrow V_{j+2} \rightarrow V_{j+3} \rightarrow \cdots$$
$$\searrow \qquad\quad \searrow \qquad \searrow \qquad \searrow$$
$$W_{j+1} \quad W_{j+2} \quad W_{j+3} \qquad \cdots$$

Consider the sequence of spaces W_j, since each is the orthogonal complement of its companion scaling space V_j and W_{j+1} is a subspace of V_j, it is clear that W_j

is orthogonal to W_{j+1}. Additionally, we saw earlier that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, this implies $\bigcup_{j\in\mathbb{Z}} W_j = L^2$. Since the W_j's are all orthogonal, we have a direct sum decomposition of L^2 namely,

$$L^2 = \bigoplus_{j\in\mathbb{Z}} W_j.$$

This certainly is helpful, as all we need to do now is find an orthogonal basis for each W_j. Taking the union of all these basis functions will yield an orthogonal basis for L^2.

This may seem to be nearly as daunting a task as we initially faced, but we can now take advantage of the highly structured nature of the V_j spaces and the orthogonality of W_j to the V_j. First, observe that since $W_j \subset V_{j-1}$, perhaps we can use the orthogonal basis for V_{j-1} to construct an orthogonal basis for W_j. Let us proceed along these lines. Any prospective basis function, φ, for W_0, must be an element of W_0 and since $W_0 \subset V_{-1}$ we must have the following representation for φ

$$\varphi = \sum_l d_l \phi(2x - l). \tag{III.4}$$

Such functions φ are the wavelets which we have alluded to throughout this paper. Since we are interested in compactly supported basis functions we assume $\{d_l\}$ to be finite. We also require φ to be orthogonal to ϕ and all its integer translates, since W_0 is the orthogonal complement of V_0. To aid us in our investigation we use the following representation of the Poisson Summation Formula, a derivation of which is found in the appendix,

$$\sum_l \langle \varphi(x), \phi(x - l) \rangle e^{-i\omega l} = \sum_k \Psi(\omega + 2k\pi)\overline{\Phi(\omega + 2k\pi)} \quad \text{for } k, l \in \mathbb{Z}. \tag{III.5}$$

Observe that this formula establishes the equivalence of the following statements:

1. $\varphi(x)$ is orthogonal to $\phi(x - l)$ for $l \in \mathbb{Z}$.

2. $\sum_{k\in\mathbb{Z}} \Psi(\omega + 2k\pi)\overline{\Phi(\omega + 2k\pi)} = 0$ for all $\omega \in \mathbb{R}$.

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We can see this since if 1 is true then the sum on the left hand side of III.5 is clearly zero for all $\omega \in \mathbb{R}$. Conversely, if 2 is true then the left hand side of III.5 is a Laurent series in $e^{i\omega}$ which is zero at all values of ω, and must be identically zero, therefore each of the inner products in the sum must be zero.

Now, let us apply this to the problem of finding a basis for W_0. Suppose $\varphi(x)$ is orthogonal to $\phi(x - k)$ for $k \in \mathbb{Z}$, this is equivalent to

$$\sum_k \Psi(\omega + 2k\pi) \overline{\Phi(\omega + 2k\pi)} = 0 \qquad \text{for all} \quad \omega \in \mathbb{R}. \tag{III.6}$$

We can use III.4 to get an expression for Ψ, the Fourier Transform of φ

$$\Psi(\xi) = \int_{-\infty}^{\infty} \sum_l d_l \phi(2x - l) e^{-ix\xi} dx = \frac{1}{2} \sum_l d_l e^{\frac{-i\xi l}{2}} \Phi\left(\frac{\xi}{2}\right).$$

As we have seen before, the Fourier Transform, Φ, of ϕ can be expressed

$$\Phi(\xi) = \frac{1}{2} \sum_n c_n e^{\frac{-i\xi n}{2}} \Phi\left(\frac{\xi}{2}\right).$$

Consider the function

$$F(\xi) = \sum_k |\Phi(\xi + 2k\pi)|^2.$$

We can write this function in the following way,

$$F(\xi) = \sum_k \Phi(\xi + 2k\pi) \overline{\Phi(\xi + 2k\pi)}.$$

Now, by III.5 F is equal to $\sum_l \langle \phi(x), \phi(x - l) \rangle = 1$ because ϕ_{0k} form an orthonormal basis for V_0. So, $F(\xi) = 1$ for all $\xi \in \mathbb{R}$.

Define the following trigonometric polynomials

$$D(\xi) = \frac{1}{2} \sum_l d_l e^{-i\xi l}$$

$$C(\xi) = \frac{1}{2} \sum_n c_n e^{-i\xi n}.$$

Substituting into III.6 we get

$$0 = \sum_k \Psi\left(\omega + 2k\pi\right) \overline{\Phi\left(\omega + 2k\pi\right)}$$

$$= \sum_k D\left(\frac{\omega + 2k\pi}{2}\right) \Phi\left(\frac{\omega + 2k\pi}{2}\right) \overline{C(\frac{\omega + 2k\pi}{2})\Phi\left(\frac{\omega + 2k\pi}{2}\right)}. \tag{III.7}$$

Observe when k is even $D\left(\frac{\omega+2k\pi}{2}\right) = D\left(\frac{\omega}{2}\right)$, and when k is odd $D\left(\frac{\omega+2k\pi}{2}\right) = D\left(\frac{\omega}{2} + \pi\right)$. Similarly, when k is even $C\left(\frac{\omega+2k\pi}{2}\right) = C\left(\frac{\omega}{2}\right)$, and when k is odd $C\left(\frac{\omega+2k\pi}{2}\right) = C\left(\frac{\omega}{2} + \pi\right)$. We can rewrite the sum as

$$\sum_k \left[D\left(\frac{\omega}{2}\right) \overline{C\left(\frac{\omega}{2}\right)} \left| \Phi\left(\frac{\omega + 2\left(2k\right)\pi}{2}\right) \right|^2 \right.$$

$$\left. + D\left(\frac{\omega}{2} + \pi\right) \overline{C\left(\frac{\omega}{2} + \pi\right)} \left| \Phi\left(\frac{\omega + 2\left(2k+1\right)\pi}{2}\right) \right|^2 \right]. \tag{III.8}$$

Factoring out the terms which are independent of k we get

$$D\left(\frac{\omega}{2}\right) \overline{C\left(\frac{\omega}{2}\right)} \sum_k \left| \Phi\left(\frac{\omega}{2} + 2k\pi\right) \right|^2$$

$$+ D\left(\frac{\omega}{2} + \pi\right) \overline{C\left(\frac{\omega}{2} + \pi\right)} \sum_k \left| \Phi\left(\frac{\omega}{2} + \pi + 2k\pi\right) \right|^2. \tag{III.9}$$

The summations above are simply $F\left(\frac{\omega}{2}\right)$ and $F\left(\frac{\omega}{2} + \pi\right)$ so we can write

$$\sum_k \Psi\left(\omega + 2k\pi\right) \overline{\Phi\left(\omega + 2k\pi\right)} =$$

$$D\left(\frac{\omega}{2}\right) \overline{C\left(\frac{\omega}{2}\right)} F\left(\frac{\omega}{2}\right) + D\left(\frac{\omega}{2} + \pi\right) \overline{C\left(\frac{\omega}{2} + \pi\right)} F\left(\frac{\omega}{2} + \pi\right). \tag{III.10}$$

But $F\left(\frac{\omega}{2}\right) = F\left(\frac{\omega}{2} + \pi\right) = 1$, so we have

$$0 = \sum_k \Psi\left(\omega + 2k\pi\right) \overline{\Phi\left(\omega + 2k\pi\right)} = D\left(\frac{\omega}{2}\right) \overline{C\left(\frac{\omega}{2}\right)} + D\left(\frac{\omega}{2} + \pi\right) \overline{C\left(\frac{\omega}{2} + \pi\right)}.$$

$$\tag{III.11}$$

To simplify our notation we will use ω in place of $\frac{\omega}{2}$. Since ϕ_{0k} form an orthonormal set, $\overline{C\left(\omega\right)}$ and $\overline{C\left(\omega + \pi\right)}$ cannot simultaneously be zero; this is a consequence of the third condition of Theorem II.1. If III.11 is to be true then when $\overline{C\left(\omega\right)}$ is zero, $\overline{C\left(\omega + \pi\right)}$

cannot be zero. Therefore, $\overline{D\left(\omega+\pi\right)}$ must be zero. Similarly, $\overline{C\left(\omega+\pi\right)}$ and $D\left(\omega\right)$ must have the same zero structure. Since $C\left(\omega\right)$ is a trigonometric polynomial, $D\left(\omega\right)$ must have the following form

$$D\left(\omega\right) = A\left(\omega\right)\overline{C\left(\omega+\pi\right)}.$$

Where $A\left(\omega\right)$ is 2π-periodic, and $A\left(\omega+\pi\right) = -A\left(\omega\right)$. Further, since we desire φ to be compactly supported, $A\left(\omega\right)$ must be a trigonometric polynomial. Now, III.11 becomes

$$
\begin{aligned}
0 &= A\left(\omega\right)\overline{C\left(\omega+\pi\right)C\left(\omega\right)} + A\left(\omega+\pi\right)\overline{C\left(\omega+\pi+\pi\right)C\left(\omega+\pi\right)} \\
&= A\left(\omega\right)\overline{C\left(\omega+\pi\right)C\left(\omega\right)} - A\left(\omega\right)\overline{C\left(\omega\right)C\left(\omega+\pi\right)}
\end{aligned}
$$

since $C\left(\omega\right)$ is 2π-periodic. Thus we have shown, subject to the above conditions on $A\left(\omega\right)$, functions generated by III.4 lie in the orthogonal complement of V_0 in V_{-1}. What remains to be shown is that these functions ψ are orthogonal to their integer translates.

To show this we will again use the Poisson Summation Formula III.5. We see that orthogonality of $\varphi\left(x\right)$ to $\varphi\left(x-l\right)$ for $l \in \mathbb{Z}$ is equivalent to

$$1 = \sum_k \Psi\left(\omega+2k\pi\right)\overline{\Psi\left(\omega+2k\pi\right)}.$$

Substituting in the expression for the Fourier Transform of φ and simplifying in a manner similar to that used above we get

$$1 = \sum_k \Psi\left(\omega+2k\pi\right)\overline{\Psi\left(\omega+2k\pi\right)} = D\left(\omega\right)\overline{D\left(\omega\right)} + D\left(\omega+\pi\right)\overline{D\left(\omega+\pi\right)}. \quad \text{(III.12)}$$

Since $D\left(\omega\right) = A\left(\omega\right)\overline{C\left(\omega+\pi\right)}$, we have

$$1 = A\left(\omega\right)\overline{C\left(\omega+\pi\right)A\left(\omega\right)}C\left(\omega+\pi\right) + A\left(\omega+\pi\right)\overline{C\left(\omega\right)A\left(\omega+\pi\right)}C\left(\omega\right).$$

Reorganizing slightly and recalling that $A\left(\omega+\pi\right) = -A\left(\omega\right)$ we find

$$1 = A\left(\omega\right)\overline{A\left(\omega\right)}C\left(\omega+\pi\right)\overline{C\left(\omega+\pi\right)} + A\left(\omega\right)\overline{A\left(\omega\right)}C\left(\omega\right)\overline{C\left(\omega\right)}.$$

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Since $\phi(x)$ is orthogonal to $\phi(x-l)$, we know $C(\omega+\pi)\overline{C(\omega+\pi)}+C(\omega)\overline{C(\omega)}=1$. If we additionally require $|A(\omega)|=1$, then III.12 will be satisfied, and $\varphi(x)$ will be orthogonal to $\varphi(x-l)$.

Let us collect the conditions we have imposed on $A(\omega)$ and see what type function they characterize.

1. $A(\omega)$ is a trigonometric polynomial, and 2π-periodic.

2. $A(\omega+\pi)=-A(\omega)$.

3. $|A(\omega)|=1$.

Condition 3 implies the graph of $A(\omega)$ traces out the unit circle or, in light of condition 1, $A(\omega)=a_0 e^{-im\omega}$ for $|a_0|=1$ and some m. Condition 2 implies m must be odd. We have a great deal of freedom in our selection of a_0, but since we have so far only considered real valued $\{d_l\}$, we will limit ourselves to $a_0=\pm1$ and for this development we will assume $a_0=1$. We will let m be the highest index of the set $\{c_k\}$[1], that is, if $\{c_k\}$ has four elements we will set $m=3$, because $\{c_k\}=\{c_0,c_1,c_2,c_3\}$.

Having made these choices we see

$$D(\omega)=e^{im\omega}\overline{\frac{1}{2}\sum_{k=0}^{m}c_k e^{-ik(\omega+\pi)}}.$$

Or, after simplifying

$$D(\omega)=\frac{1}{2}\sum_{k=0}^{m}(-1)^k\overline{c_k}e^{i(k+m)\omega},$$

which we can express as

$$D(\omega)=\frac{1}{2}\sum_{l=0}^{m}d_l e^{-il\omega},$$

---

[1]Condition II.11 implies the highest index of $\{c_k\}$ must be odd. Further any odd m gives rise to a translated version of this same φ.

where $d_l = (-1)^{m-l}\,\overline{c_{m-l}}$. That is $\{d_l\} = \{-\overline{c_m}, \overline{c_{m-1}}, -\overline{c_{m-2}}, \cdots, \overline{c_2}, -\overline{c_1}, \overline{c_0}\}$. Recall this $\{d_l\}$ is such that $\varphi(x) = \sum_l d_l \phi(2x - l)$ and its integer translates $\varphi(x - k)$ form an orthonormal set in W_0.

It is not obvious that this set spans W_0. We can see that it does by employing an argument similar to that above. Suppose $g(x) \in W_0$, then $g(x) \in V_{-1}$ and $g(x) \perp V_0$. We can express $g(x)$ as

$$g(x) = \sum_l \gamma_l \phi(2x - l).$$

This, in turn gives us $G(\xi)$, the Fourier Transform of $g(x)$

$$G(\xi) = \frac{1}{2} \sum_l \gamma_l e^{-\frac{il\xi}{2}} \Phi\left(\frac{\xi}{2}\right).$$

Define the Laurent series in $e^{-i\xi}$, $\Gamma(\xi)$ by

$$\Gamma(\xi) = \frac{1}{2} \sum_l \gamma_l e^{-il\xi}.$$

Since $g(x) \perp V_0$, we can obtain the following relationship between $C(\xi)$ and $\Gamma(\xi)$

$$0 = \Gamma\left(\frac{\xi}{2}\right) \overline{C\left(\frac{\xi}{2}\right)} + \Gamma\left(\frac{\xi}{2} + \pi\right) \overline{C\left(\frac{\xi}{2} + \pi\right)}.$$

Now, by exploiting the zero structure of $C(\xi)$ we can see $\Gamma(\xi)$ must have the following form

$$\Gamma(\xi) = B(\xi) \overline{C(\xi + \pi)},$$

where $B(\xi)$ satisfies the following:

1. $B(\xi)$ is a Laurent series in $e^{-i\xi}$.

2. $B(\xi)$ is 2π-periodic.

3. $B(\xi + \pi) = -B(\xi)$.

Conditions 2 and 3 above imply $B(\xi)$ consists solely of odd powers of $e^{-i\xi}$. Each of these odd powers in turn corresponds to a translation of the function φ. So,

any function $g(x) \in W_0$ can be represented by a weighted sum of $\varphi(x - k)$. Thus we have shown $\varphi(x - k)$ span the space W_0.

To summarize, so far we have a sequence of spaces W_j the direct sum of which is L^2. We have demonstrated a basis $\{\varphi(x - k)\}$ for W_0 which is constructed from functions $\phi(2x - l)$ which form a basis for V_{-1}. Now we need to show an orthonormal basis for W_j is given by $\{\varphi_{jk} = 2^{-j/2}\varphi(2^{-j}x - k)\}$. To do this we use III.5. From standard Fourier Transform operations we can see

$$\Psi_{jk}(\xi) = 2^{j/2}e^{-2^jki\xi}\Psi(2^j\xi)$$
$$\Phi_{jk}(\xi) = 2^{j/2}e^{-2^jki\xi}\Phi(2^j\xi).$$

It is easily verified using arguments similar to those above that φ_{jk} is orthogonal to ϕ_{jk}. It is equally easy to verify $\langle \varphi_{jk}, \varphi_{jn} \rangle = \delta_{n,k}$. Thus for each scaling level j we have an orthonormal set $\{\varphi_{jk}\}$, which lies in W_j. All that remains to show is that each set spans the respective space W_j. This is accomplished in a manner completely similar to that used for W_0. Finally, we have an orthonormal basis for L^2, namely, $\{\varphi_{jk} = 2^{-j/2}\varphi(2^{-j}x - k)\}$ for $j, k \in \mathbb{Z}$.

B.   ALGORITHM DEVELOPMENT

Armed with this basis for L^2 we now set out to implement a discrete transformation which, as promised in the introduction, has good time-frequency localization, suitable power of approximation, and can be computed quickly. Time localization is a result of the compact support of the scaling functions and wavelets, and the fact that the support of these functions shrinks as the index of the subspaces V_j and W_j decrease (toward $-\infty$). Frequency localization is accomplished by dividing the spectrum of the input function into "octaves" each of which is represented by the projection of input function onto some W_j. This is not as sharp as the frequency localization of the Fourier Transform, but it is still quite good, and can be computed rapidly. The entire derivation of the basis functions was driven by the desire to take advantage of a recursive relation between basis functions. We do not have a recursive

relation between the actual basis functions φ_{jk}, however, we do have a linear relation between the basis functions and a set of auxiliary functions ϕ_{jk}, which we call the scaling functions

$$\varphi(x) = \sum_l d_l \phi(2x - l).$$
(III.13)

This would not be of much use without the associated recursive relation between the scaling functions at subsequent levels of scaling

$$\phi(x) = \sum_k c_k \phi(2x - k).$$
(III.14)

We will exploit these relations to achieve a computational efficiency of order N operations. In computing a transform we are concerned with projections. Since projection operators are linear, we can recast III.15 and III.16 in terms of the projection of an arbitrary function $f(x)$ on ϕ_{jk} and φ_{jk}. Specifically,

$$\langle f(x), \phi(x) \rangle = \sum_k c_k \langle f(x), \phi(2x - k) \rangle$$
(III.15)

and

$$\langle f(x), \varphi(x) \rangle = \sum_l d_l \langle f(x), \phi(2x - k) \rangle.$$
(III.16)

Put another way, these relationships allow us to compute the projection of $f(x)$ onto W_j given the projection of $f(x)$ onto V_{j-1}.

Before we can develop an algorithm, we need to recognize that the theory we have developed so far defines a transform which operates on functions in L^2, square summable functions defined on the real numbers. Further, the transform projects the input function onto a bi-infinite sequence of subspaces of L^2. In practice we are given a sequence of uniformly sampled values of the input function instead of the actual function. We will, for the purposes of this development, assume the input function f is zero at its boundaries. By doing this we can "pad" the function with zeros and apply the transform as if the function were defined over the entire real

line. There are methods of treating input functions which are not zero at their boundaries, but these methods are beyond the scope of this development. Next we must address the question of where in the bi-infinite sequence of subspaces to start the transform. The sampled values of f are a piecewise constant approximation to the actual input function[2]. We will define the <u>fine scaling unit</u> to be the "distance" (difference in independent variable) between successive samples. A consequence of the orthogonality of the scaling functions is that they must have approximation power at least one. This is easily seen since $|m_0(z)|^2 + |m_0(z+\pi)|^2 = 1$ for all $z \in \mathbb{R}$, and $m_0(0) = 1$, $m_0(\pi) = 0$. Put another way, the polynomial with coefficients $\{c_k\}$ has a root at π, which tells us the scaling function associated with $\{c_k\}$ has approximation power at least one. How do we use this fact? Over the region where f is represented by a constant, the scaling function and its translates can exactly represent f. Thus the projection of f onto the wavelets, φ_{jk}, at this scaling level is zero. Also, at any finer scaling level the projection of f onto the wavelets must be zero. In light of these facts we will start the algorithm at scaling level l, where the scaling function $2^{-l/2}\phi(2^l x)$ has support width of one fine scaling unit. We call this the finest scaling level.

We now outline the algorithm to compute the discrete, orthogonal wavelet transform. Figure 3 depicts the arrangement of wavelet and scaling functions. The first step in the algorithm is to calculate the projection of the input function onto the scaling functions at the finest scaling level. There are many ways to accomplish this, one way is presented in the appendix. Once we have these projections, the algorithm is simply a matter of using III.15 and III.16 to reorganize the data into the projections onto V_{l+1} and W_{l+1}. This is accomplished by taking weighted sums of the projections onto V_l. To calculate projections onto V_{l+1}, we use as weights $\{c_k\}$. To calculate the

---

[2]Other approximations may be used, for instance, if the input function is represented in a B-spline basis of order two, we would have a piecewise quadratic approximation to the input function. Similar results for higher order approximations can be developed in a manner similar to that used for piecewise constant approximation.

projections onto W_{l+1}, we use as weights $\{d_\lambda\}$. We repeat this process until we have calculated the projections onto all of the subspaces V_j and W_j, for $l \leq j \leq m$. Here m is the index such that the scaling function $2^{-m/2}\phi(2^m x)$ has width of support equal to that of f. The discrete, orthogonal wavelet transform of the function f is given by the following collection of projections

$$\mathcal{W}(f) = \{\langle f, \varphi_{jk} \rangle \mid l \leq j \leq m\} \cup \{\langle f, \phi_{mk} \rangle\}.$$

The following is a pseudocode version of what we have just described.
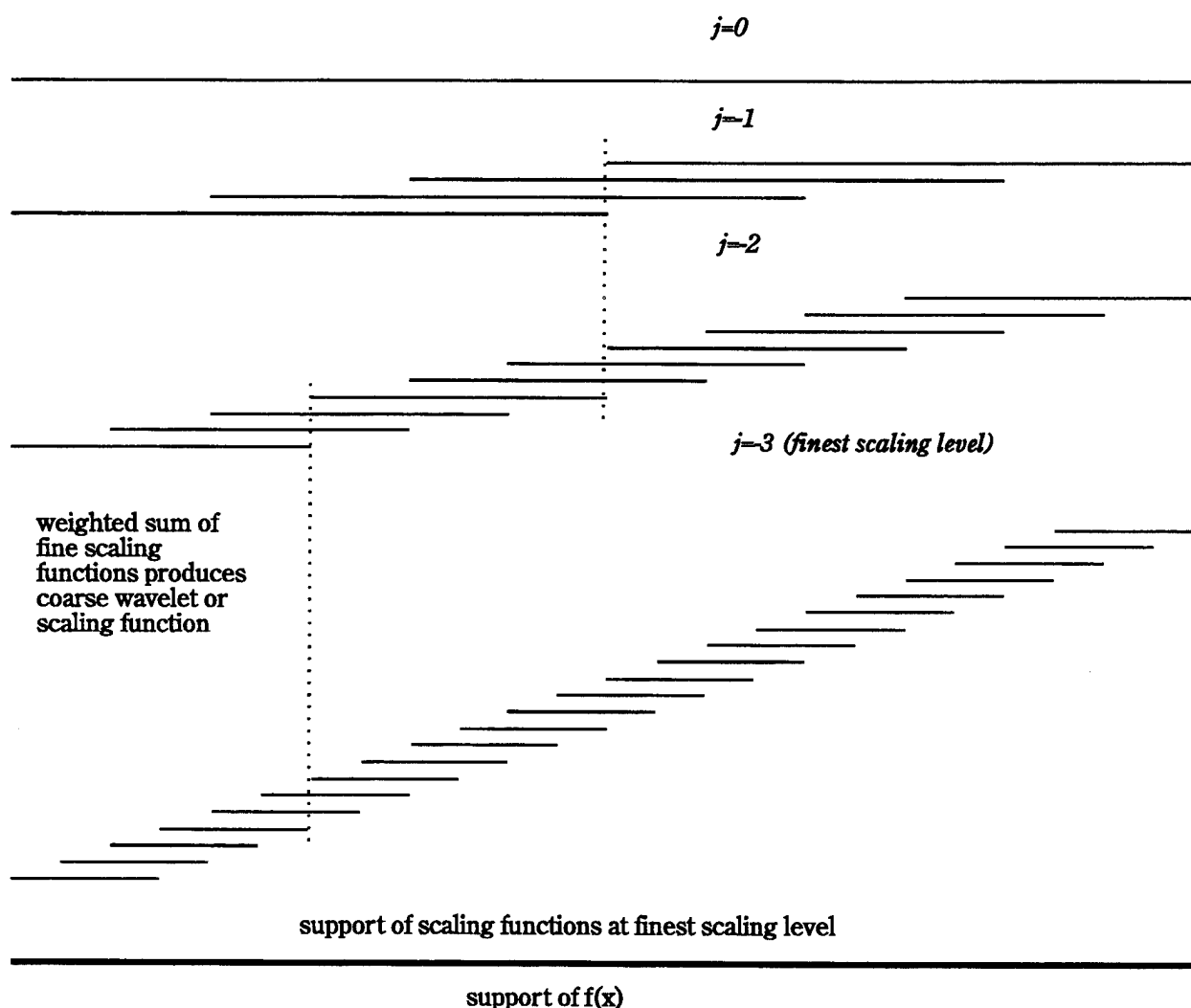
j=0

j=1

j=2

j=3 (finest scaling level)

weighted sum of
fine scaling
functions produces
coarse wavelet or
scaling function

support of scaling functions at finest scaling level

support of f(x)

Figure 3. Organization of Basis Function

```
input:
```

1. $n + 1$ dilation equation coefficients for the wavelet we wish to use
2. sequence of N values which represent the projection of f onto ϕ_{lk} at the finest scaling level. $N = n2^{p-1} + 1$ for some p.

```
begin algorithm:
for j = l - m - 1 : 0
for k = 0 : n2^{j-1} - 3
```
$$\langle f, \phi_{jk} \rangle = \sqrt{2} \sum_{l=2k}^{2k+n} c_{l-2k} \langle f, \phi_{j-1,l} \rangle$$
$$\langle f, \varphi_{jk} \rangle = \sqrt{2} \sum_{l=2k}^{2k+n} c_{l-2k} \langle f, \phi_{j-1,l} \rangle$$
```
end
end
```
```
output:
```
$\mathcal{W}(f) = \{ \langle f, \varphi_{jk} \rangle \mid l \leq j \leq 0 \} \cup \{ \langle f, \phi_{0k} \rangle \}$

This brings up another aspect of implementing the wavelet transform on a computer. The sequence of subspaces we project onto is bi-infinite. We saw earlier that we could pick a starting index based on the resolution with which f was represented. The algorithm to compute the transform must stop at some index. What should this index be, and what are the consequences of stopping the algorithm here? In this development we will stop the transform at the index where the scaling function has support width equal to the width of the input function sample. At each step of the transform we are calculating the projections of f onto succeedingly lower frequency basis functions as the support width of φ_{jk} increases. Terminating the algorithm at any given index establishes a low frequency bound below which we have no frequency localization.

This section has presented a very basic outline of the discrete, orthogonal wavelet transform algorithm. There are many choices as to the indices on which the algorithm should begin and terminate. The major concept of the algorithm is the use of the relationships III.15 and III.16 to rapidly compute projections at one level from those at a preceding level. It is precisely these relationships which allow the wavelet transform to be computed in order N operations as we shall now see.

To start our study of the computational efficiency of the algorithm we will assume the input function is given as N sampled values. We further assume that $N = 2^p n + 1$ for some p where n is the highest index of $\{c_k\}$. We divide the algorithm into the following blocks.

1. Calculation of projections at finest scaling level: Nearly any quadrature rule can be used to calculate these projections in $O(N)$ operations.

2. Calculation of wavelet coefficients: At each level j of the p levels of the algorithm we must calculate $n2^j - 2$ wavelet coefficients. each of these coefficients is calculated by taking a weighted sum of projections onto the scaling coefficients at the preceding level. The computational cost for the weighted sum is $n + 1$ multiplies and n additions. this makes the total cost for this block

$$\sum_{j=0}^{p-1} (n2^j - 2)(2n + 1).$$

3. Calculation of scaling function coefficients : At each level we must calculate the projections of the function onto the scaling functions. This is accomplished by taking a weighted sum of projections onto the scaling functions at the preceding level. The computational cost is identical to that for calculating the wavelet coefficients above.

If we sum the computational cost for each of these blocks we get a total algorithmic cost of

$$O(N) + 2\sum_{j=0}^{p-1} (n2^j - 2)(2n + 1).$$

The above sum is dominated by $2(2n + 1)2^p$ which is $O(N)$. So, we have developed a transform which provides good time-frequency localization, can be used as an approximation scheme, and is rapidly computed. Some applications and directions for further studies are discussed in the next chapter.

IV. CONCLUSIONS AND DIRECTIONS FOR FURTHER STUDY

We have shown that we can develop a transform based on solutions to the dilation equation. This transform is useful because we can alter the characteristics of this transform simply by changing the dilation equation coefficients. Also, we can compute this transform quickly. In fact, the order N operations required for the wavelet transform is considerably better than the $N \log(N)$ operations required for the fast Fourier Transform for large N. The wavelet transform has the added advantage of time localization. A major advantage of the Fourier Transform is that the basis functions are eigenfunctions for an operator of fundamental importance, namely the Laplacian operator. The wavelet transform, on the other hand, has been restricted predominantly to use in the signal processing field. In this field the wavelets time-frequency localization and rapid computation are great assets, and projection onto a basis of eigenfunctions is not always as desirable as these other properties.

To be as useful as the Fourier Transform, wavelets must be "well behaved" under some operator of interest. By well behaved we mean the image of the wavelet under the operator is simply related to the wavelet or its translations and dilations. Eigenfunctions for an operator are the ultimate in well behaved functions. It is clear from our preliminary work in Chapter II that convolution and convolution type operations are the natural operations for wavelets. If we are to find an operator under which wavelets would be well behaved, it is natural to first consider operators based on convolution like processes. A broad field of such operators is the Zygmund-Calderon group of operators. A description of these operators is beyond the scope of this work, but this is a burgeoning area for wavelet analysis.

An area which has great potential is that of tailoring a wavelet to be well behaved under a particular operator. To see how we might tailor a wavelet we should consider that if we wish our wavelet to have support of width n at the zeroth scaling

level, $\{c_k\}$ must have $n+1$ elements, namely $c_0, \dots, c_n$. We still desire the wavelets to be orthogonal, so we must meet the necessary conditions of section I.3. Specifically, $\sum_k c_k \overline{c_{k-2m}} = 2\delta_m$ for $m \in \mathbb{Z}$. Each of these conditions can be represented as a simple bi-linear form

$$\mathbf{c}^* \mathbf{D_{2m}} \mathbf{c} = 2\delta_{\mathbf{m}},$$

where $*$ indicates conjugate transposition and $\mathbf{D_{2m}}$ is the $2m$ down shift matrix. That is, $\mathbf{D_{2m}}$ has ones on its $2m^{th}$ diagonal and zeros elsewhere. If these conditions were linear then we could easily determine the solution space for this system. However, these conditions are not linear and systems of bi-linear forms are not well studied. We can see for $m = 0$ the condition is $\sum_k |c_k|^2 = 2$. We can also see for each positive value of m the condition is equivalent to the corresponding condition for $-m$, since $\mathbf{c}^* \mathbf{D_{2m}} \mathbf{c} = 0 = 0^* = \mathbf{c}^* \mathbf{D_{2m}^*} (\mathbf{c}^*)^* = \mathbf{c}^* \mathbf{D_{-2m}} \mathbf{c}$. So, we have reduced the number of conditions from n to $\frac{n+1}{2}$. To make further progress we need to be able to determine the solution space of a system of bi-linear forms. The use of the term space here refers to the set of all solutions, not necessarily a linear space. The solutions of a system of bi-linear forms need not be linear. Once we have the solutions to this system we would then determine which of these solutions could be written as linear combinations of the vectors $\left\{ \left[v_p^k \right] \right\}$ as defined in section I.2 where p is our desired approximation power. This would satisfy the sufficient condition for approximation power. Now we check these solutions against the sufficient condition for orthogonality presented in section I.3. The result would be a collection of dilation equation coefficients which would yield orthogonal wavelets with the desired approximation power. Perhaps among this collection we would find dilation equation coefficients which would be well behaved under the operator of interest.

REFERENCES

[1] Volkmer, H., *Distributional and Square Summable Solutions of Dilation Equations* Preprint

[2] Daubechies, I., *Ten Lectures on Wavelets*, SIAM Philadelphia PA 1992

[3] Strang G. and Fix, G. *A Fourier Analysis of the Finite Element Variational Method*, Centro Internazionale Matematico Estivo, Constructive Aspects of Functional Analysis, Edizioni Cremonese, Roma 1973.

[4] Schoenberg, I.J., *Cardinal Spline Interpolation* SIAM Philadelphia PA, 1973

[5] Cohen, A., *Ondelettes, Analyses Multiresolutions et Filtres Miroir en Quadrature*, Ann. Inst. H. Poincare, Anal. Non Lineaire, 7, pp. 439-459, 1990

[6] Royden, H.L., *Real Analysis*, Macmillian Publishing Company, New York NY, 1988

APPENDIX A. DERIVATION OF POISSON SUMMATION FORMULA

We start with the Poisson Summation Formula as found in any text on Fourier Analysis

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} F(2k\pi),$$

where $F(\xi)$ is the Fourier Transform of $f(x)$. Consider the function

$$f(\omega, l) = \langle \psi(x), \phi(x-l) \rangle e^{-i\omega l}.$$

We see the Fourier Transform, in the variable l, of f is given by

$$F(\omega, \xi) = \int_{-\infty}^{\infty} \langle \psi(x), \phi(x-l) \rangle e^{-i\omega l} e^{-i\xi l} dl.$$

Expanding the inner product and simplifying the exponential we get

$$F(\omega, \xi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x) \overline{\phi(x-l)} dx \; e^{-il(\omega+\xi)} dl.$$

Now, change the order of integration, and we see

$$F(\omega, \xi) = \int_{-\infty}^{\infty} \psi(x) \int_{-\infty}^{\infty} \overline{\phi(x-l)} \; e^{-il(\omega+\xi)} dl \; dx.$$

Make the change of variables $y = x - l$, and we now have

$$F(\omega, \xi) = \int_{-\infty}^{\infty} \psi(x) e^{-ix(\omega+\xi)} \int_{-\infty}^{\infty} \overline{\phi(y)} \; e^{iy(\omega+\xi)} dy \; dx.$$

We can write the integral in y above as

$$\int_{-\infty}^{\infty} \overline{\phi(y)} \; e^{iy(\omega+\xi)} dy = \overline{\Phi(\omega+\xi)}.$$

This leaves us with

$$F(\omega, \xi) = \int_{-\infty}^{\infty} \psi(x) e^{-ix(\omega+\xi)} \overline{\Phi(\omega+\xi)} dx.$$

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Since $\overline{\Phi\left(\omega+\xi\right)}$ has no x dependence, we can write this as

$$F\left(\omega,\xi\right)=\int_{-\infty}^{\infty}\psi\left(x\right)e^{-ix\left(\omega+\xi\right)}dx\overline{\Phi\left(\omega+\xi\right)}.$$

The integral in x above is simply the $\Psi\left(\omega+\xi\right)$ which leads us to

$$F\left(\omega,\xi\right)=\Psi\left(\omega+\xi\right)\overline{\Phi\left(\omega+\xi\right)}.$$

Now, applying the Poisson Summation Formula we get

$$\sum_{l\in\mathbb{Z}}\left\langle\psi\left(x\right),\phi\left(x-l\right)\right\rangle e^{-i\omega l}=\sum_{k\in\mathbb{Z}}\Psi\left(\omega+2k\pi\right)\overline{\Phi\left(\omega+2k\pi\right)},$$

which is the desired representation of the Poisson Summation Formula.

APPENDIX B. PROJECTIONS AT FINEST SCALING LEVEL

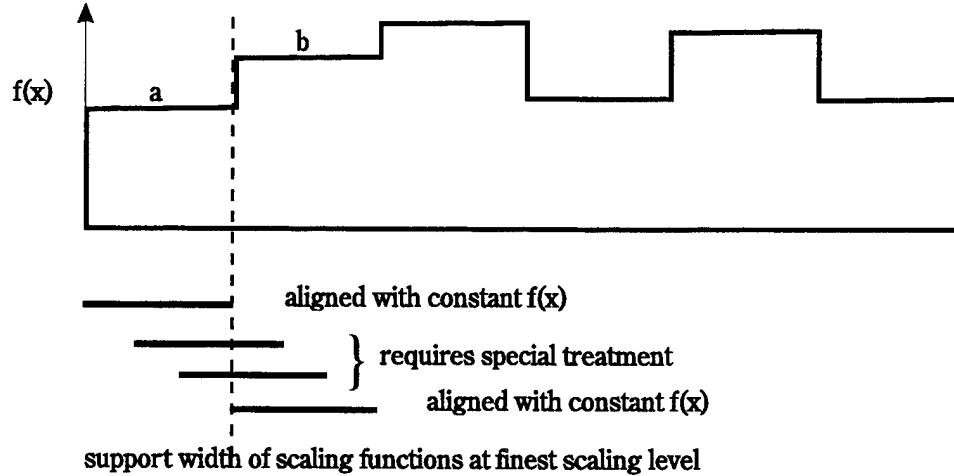To see how we might accomplish this, consider the following diagram.

Figure 4. Scaling Functions at Finest Scaling Level

Each projection at the finest scaling level is the projection of ϕ_{lk} onto either a constant or piecewise constant function. Where f is constant over the support of ϕ_{lk} the projection is simply a weighted integral of ϕ_{lk}. Where f is piecewise constant over the support of ϕ_{lk} we can divide the domain of integration so that the projection is given by a sum of integrals over domains where f is constant. Each of these sub-integrals is given by a weighted integral of ϕ_{lk} over a portion of its support. For instance, if we are computing the projection of f onto the second scaling function in Figure 4, we would have the following

$$\int_0^3 f(x)\,\phi_2(x)\,dx = a \int_0^1 \phi_2(x)\,dx + a \int_1^2 \phi_2(x)\,dx + b \int_2^3 \phi_2(x)\,dx.$$

Similarly, the projection onto the third scaling function would be

$$\int_0^3 f(x)\,\phi_3(x)\,dx = a \int_0^1 \phi_3(x)\,dx + b \int_1^2 \phi_3(x)\,dx + b \int_2^3 \phi_3(x)\,dx.$$

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The sub-integrals, $\int_0^1 \phi_3(x)\, dx$, can be computed once and stored for use since they depend only on the wavelet used. We will assume the projection of the input function onto W_l is zero. This prevents us from incorporating into the transform the discontinuities introduced by the piecewise constant approximation of f. It is worth noting that at this point in the algorithm we have all the information we ever will about the input function. In fact, we do not use the function for the rest of the algorithm. The remainder of the algorithm is devoted to rearranging the projections of f onto the scaling functions at the finest level into projections of f onto the spaces W_j for $j > l$.

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